ON THE NUMBER OF GENERATORS OF IDEALS DEFINING GORENSTEIN ARTIN ALGEBRAS WITH HILBERT FUNCTION $(1, n+1, 1+\binom{n+1}{2}, \ldots, \binom{n+1}{2}+1, n+1, 1)$

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Abstract. Let $R = k[w, x_1, \ldots, x_n]/I$ be a graded Gorenstein Artin algebra . Then $I = \operatorname{ann} F$ for some F in the divided power algebra $k_{DP}[W, X_1, \ldots, X_n]$. If RI_2 is a height one ideal generated by n quadrics, then $I_2 \subset (w)$ after a possible change of variables. Let $J = I \cap k[x_1, \ldots, x_n]$. Then $\mu(I) \leq \mu(J)+n+1$ and I is said to be generic if $\mu(I) = \mu(J) + n + 1$. In this article we prove necessary conditions, in terms of F, for an ideal to be generic. With some extra assumptions on the exponents of terms of F, we obtain a characterization for $I = \operatorname{ann} F$ to be generic in codimension four.

Introduction

Let $R = k[x_1, \ldots, x_r]$ and I be a homogeneous ideal of R. Let $A = R/I$ be an Artinian Gorenstein quotient of R. If $r = 2$, then it is known that I is a complete intersection. If $r = 3$, then by Buchsbaum and Eisenbud structure theorem [2], I is the $(2n)^{th}$ order pfaffians of a $(2n+1)^{th}$ order skew symmetric matrix. When $r = 4$, Kustin and Miller gave a structure theorem for Gorenstein Artinian ideals of the form (f, g, h, x_4J), where J is a Gorenstein ideal of height three. Let $I = \oplus_{n\geq 1} I_n$ be the direct sum decomposition of a Gorenstein Artinian ideal in $k[x, y, z, w]$. In [4], Iarrobino and Srinivasan studied several properties of the Gorenstein ideals I such that $I_2 \cong \langle wx, wy, wz\rangle$ or $I_2 \cong \langle wx, wy, w^2\rangle$. They gave a structure theorem for ideals I with $I_2 = \langle wx, wy, wz\rangle$ and Hilbert function $H_{R/I} = (1, 4, 7, \ldots, 1)$. They did this by establishing a connection between some properties of the ideal I and $J = I \cap R'$, where $R' = k[x, y, z]$, which is a height three Gorenstein ideal. When $I_2 = \langle wx, wy, w^2\rangle$, they named these algebras *mysterious Gorenstein algebras* and studied their various properties. They showed that such an ideal can be obtained as an annihilator of a homogeneous form $F = G(X, Y, Z) + WZ^{[j-1]} \in k_{DP}[X, Y, Z, W]$, where $k_{DP}[X, Y, Z, W]$ denotes the divided power algebra. They studied the Hilbert function properties of R/I connecting it with those of R'/J.

In [3], El Khoury and Srinivasan studied certain properties of Gorenstein Artinian algebras of the form R/I, where $I_2 = \langle wx, wy, w^2\rangle$. They gave a structure theorem for

such ideals. They showed that I is generated by I_2, $J = I \cap R'$ and an element of the form $wz^\beta - g$, where $g \in R' \notin I$.

It can easily be seen that $gx, gy \in J$. El Khoury and Srinivasan proved that $\{wx, wy, w^2, \alpha_1, \ldots, \alpha_{n-1}, z^j, wz^\beta - g\}$ is a minimal generating set for I unless gx or gy is already a minimal generator of $J = (\alpha_1, \ldots, \alpha_{n-1}, z^j)$. In that case, dropping each of gx and gy that is a minimal generator for J, the remaining $n+3$ or $n+2$ elements minimally generate I. They obtained a minimal free resolution of R/I in all three cases where neither gx nor gy is a part of a minimal generating set for J, either one of them is a minimal generator and both are minimal generators for J. They also studied a special case of these ideals namely, $F = X^{[a]}H(Y, Z) + WZ^{[j-1]}$. In that case, they described completely the minimal free resolution of R/I. They concluded their article with an interesting question on classification of ideals I with $n + 2$, $n + 3$ and $n + 4$ number of generators where n denotes the minimum number of generators of J.

In this article, we consider the problem in higher embedding dimensions and partially answer this question.

Let $R = k[w = x_0, x_1, \ldots x_n]$ be a standard graded ring of dimension $n + 1$ and I a homogeneous Gorenstein ideal of height $n+1$ with $H(R/I) = (1, n+1, 1+\binom{n+1}{2}, \ldots, \binom{n+1}{2}+1, n+1, 1)$. Suppose the ideal RI_2 has height one. Then after a possible change of variables, I equals $\operatorname{ann} F$, where $F = G(X_1, \ldots, X_n) + W^{[j]}$ or $F = G(X_1, \ldots X_n) + WX_n^{[j-1]}$ under Macaulay equivalence.

Let $J = I \cap k[x_1, \ldots x_n] = I \cap R'$ so that $R = R'[w]$. If $F = G(X_1, \ldots X_n) + W^{[j]}$, then J is Gorenstein of height n and I is minimally generated by $J, wx_i, 1 \leq i \leq n-1, w^\beta - g$ for some $g \in R'$ and not in I. If $F = G(X_1, \ldots X_n) + WX_n^{[j-1]}$, then I is generated by $J, wx_i, 1 \leq i \leq n-1, w^2, wx_n^\beta - g$ for some $\beta \leq j-1$ and $g \in R' \backslash J$. In the generic case, all of these are minimal generators, that is gx_i is not a minimal generator of J for any i. Thus, if G is sufficiently general, $\mu(I) = \mu(J) + n + 1$ and we say G or $I = \operatorname{ann}(G + WZ^{[j-1]})$ is *generic*. Since there is a one-to-one correspondence between the height four Gorenstein ideals I with the property that $I_2 \cong (wx_i, 1 \leq i \leq n-1, w^2)$ and homogeneous forms $F = G(X_1, \ldots X_n) + WZ^{[j-1]} \in k_{DP}[W, X_1, \ldots X_{n-1}, X_n = Z]$, classifying such ideals is equivalent to classifying these homogeneous forms in the divided power algebra. Therefore, we try to classify the property of I being generic in terms of certain properties of the homogeneous form F such that $I = \operatorname{ann} F$.

In Section 1, we begin by comparing $I = \operatorname{ann}(G+WZ^{[j]})$ and $I_{x_i} = \operatorname{ann}(\frac{\partial G}{\partial X_i}+WZ^{[j-1]})$ in general and show that when X_i divides G, I can be generic only if I_{x_i} is generic. When

$n = 3$, for Gorenstein Artinian algebras, we have more specific results. We prove that if $I = \operatorname{ann}(F)$ is generic, then there are some relations among the X, Y and Z-degrees of F.

In Section 2, we study the case when $F = G(Y,Z) + X^{[a]}G_1 + WZ^{[j-1]}$. When $a > j/2$ we prove that I is generic if and only if $\max\deg_Y(G) = \max\deg_Y(G_1)$. In the next section, we discuss the general case, where $F = G(Y,Z) + X^{[a_1]}G_1(Y,Z) + \cdots + X^{[a_n]}G_n(Y,Z) + WZ^{[j-1]}$. Assuming that $j - a_n < a_1 < \cdots < a_n$, we obtain necessary and sufficient conditions for I to be generic.

We conclude our article by comparing the Hilbert functions of R/I and R'/J. As a consequence, we show that certain classes of Cohen-Macaulay height three ideals in $k[x,y,z]$ are unimodal, even though they are not Gorenstein.

1. Genericity

Let R be a standard graded k algebra of dimension $n+1$ and I be a graded Gorenstein ideal of height $n+1$ such that $H(R/I) = (1, n+1, 1+\binom{n+1}{2}, \ldots, n+1, 1)$. Then the ideal I has n quadrics amongst its minimal generators. Suppose these quadrics generate an ideal I_2 of height one, then there exists a one form w such that $I_2 \subset (w)$ and $R = k[w, x_1, \ldots x_n]$ for some suitable one form x_i. Let $R' = k[x_1, \ldots, x_n]$ and $J = I \cap R'$.

We will make use of the Macaulay equivalence between graded Gorenstein ideals of socle degree j in $R = k[w, x_1, \ldots x_n]$ and j-forms in the divided power algebra $R^* = k_{DP}[X_1, \ldots, X_n]$ by the action of R on R^* by differentiation. If $F \in R^*$, then $\operatorname{ann} F = \{f \in R \mid (\partial/\partial f)(F) = 0\}$.

The multiplication in the divided power algebra is different from the usual polynomial algebra : $X^{[a]} \cdot X^{[b]} = \frac{(a+b)!}{a!b!} X^{[a+b]}$.

In our situation, $I = \operatorname{ann} F$, where $F = G + W^{[j]}$ or $G + WX_n^{[j-1]}$ depending on whether $w^2 \notin I$ or $w^2 \in I$. When G is generic among $R[X_1, \ldots X_n]$, $\mu(I) = \mu(J) + n + 1$. We say that F is *generic mod* W and the corresponding ideal $I = \operatorname{ann} F$ is *generic* if $\mu(I) = \mu(J) + n + 1$.

If $w^2 \notin I$, then we can see that $I_2 = (wx_i, 1 \le i \le n)$ and R'/J is a Gorenstein Artin algebra of embedding dimension n. The minimal number of generators for I, $\mu(I)$ is precisely $n+1$ more than that of J for $I = (J, wx_1, \ldots, wx_n, w^j - g)$ for some $g \in R', g \notin J$. Thus $\mu(I) = \mu(J) + n + 1$ and in this case, I is always *generic* in our sense.

From now on, we consider the case where $w^2 \in I$. Then w^2 is one of the quadrics minimally generating I and for a suitable choice of one forms $x_1, \ldots x_{n-1}, x_n = z$, $I_2 \cong$

$(w^2, wx_1, \dots wx_{n-1})$. There exists a unique $F = G(X_1, \dots, X_{n-1}, Z) + WZ^{[j-1]}$ of degree j in the divided power algebra such that $I = \operatorname{ann} F$. It has been proved in [3] that I is generated by $(I_2, J, wz^\beta - g)$ with $J = I \cap k[x_1, \dots, x_n = z]$, $g \in k[x_1, \dots, x_{n-1}, z] \setminus J$ and $\beta \leq j-1$. Suppose J is minimally generated by $\alpha_1, \dots, \alpha_n = z^j$. Since wx_i are in I, $gx_i, 1 \leq i \leq n-1$ belong to J. Therefore, if $gx_i \in \mathfrak{n}J$, where $\mathfrak{n} = (x_1, \dots x_{n-1}, z) \subset k[x_1, \dots, x_{n-1}, z]$, then they are not part of any minimal generating set for J and hence $\mu(I) = \mu(J) + n + 1$. Depending on whether gx_i's are part of a minimal generating set for J, the number of minimal generators for I will be $\mu(J) + t$ for $2 \leq t \leq n+1$. We summarize this as:

Remark 1.1. *If $A = R/I$ is a Gorenstein Artin algebra with Hilbert function $(1, n+1, 1 + \binom{n+1}{2}), \dots, n+1, 1)$, then after a linear change of variables, $R = k[x_1, \dots, x_n, w]$ and there exists a unique minimal generator for I of the form $wx_n^t - g(x_1, \dots x_n)$. If $J = I \cap k[x_1, \dots, x_n]$, then $\mu(I) = \mu(J) + n + 1 - r$, where r is the cardinality of $\{i \mid gx_i \in J/\mathfrak{m}J\}$.*

Thus, I is generic if and only if $gx_i \in (x_1, \dots x_n)J$ for all $1 \leq i \leq n-1$. The purpose of this paper is to classify these polynomials F that give rise to generic ideals I.

We begin by proving a result which helps us restrict our study to a simpler class of polynomials. Let $F = G(X_1, \dots, X_{n-1}, Z) + WZ^{[j-1]}$ and $I = \operatorname{ann} F$. Without loss of generality, we may also assume that G does not have a term in LZ^{j-1} where L is a linear form for in that case we can replace W by $(W+L)$. In what follows we will take $I = \operatorname{ann} F$ with F as above. For any form $F \in k_{DP}[X_1, \dots, X_{n-1}, Z, W]$, F_X denotes the partial with respect to X.

Theorem 1.2. *Suppose $F' = X_t G + WZ^{[j]}$ for some t, $1 \leq t \leq n-1$. If $\operatorname{ann} F$ is generic, then so is $\operatorname{ann} F'$.*

Proof. Let $X_t = X$, $I' = \operatorname{ann} F'$ and $I = \operatorname{ann} F$. Let $R' = k[x_1, \dots, x_{n-1}, z]$ and $J' = I' \cap k[x_1, \dots, x_{n-1}, z]$. Then $I' = (wx_i, 1 \leq i \leq n-1, w^2, J', wz^{\beta'} - g')$ for some $g' \in R' \setminus J'$ and $\beta' \leq j$. We first show that $wz^{\beta'} - g'$ can be replaced with $wz^{\beta+1} - h$ for some $h \in R' \setminus J'$ and $hx \in \mathfrak{n}J'$.

Claim: $wz^\beta - g \in I$ for some g if and only if $wz^{\beta+1} - gx \in I'$.

Proof of the claim: If $wz^\beta - g \in I$, then

$$\begin{aligned} 0 &= \frac{\partial F}{\partial wz^\beta - g} = \frac{\partial G}{\partial wz^\beta} + \frac{\partial WZ^{[j-1]}}{\partial wz^\beta} - \frac{\partial G}{\partial g} - \frac{\partial WZ^{[j-1]}}{\partial g} \\ &= Z^{[j-\beta-1]} - \frac{\partial G}{\partial g} - W\frac{\partial Z^{[j-1]}}{\partial g}. \end{aligned}$$

This implies $\dfrac{\partial Z^{[j-1]}}{\partial G} = 0$ and $\dfrac{\partial G}{\partial g} = Z^{[j-\beta-1]}$. Now, consider

$$\begin{aligned} \frac{\partial(G' + WZ^{[j]})}{\partial wz^{\beta+1} - gx} &= \frac{\partial G'}{\partial wz^{\beta+1}} + \frac{\partial WZ^{[j]}}{\partial wz^{\beta+1}} - \frac{\partial G'}{\partial gx} - \frac{\partial WZ^{[j]}}{\partial gx} \\ &= Z^{[j-\beta-1]} - \frac{\partial G}{\partial g} \\ &= 0. \end{aligned}$$

Therefore $wz^{\beta+1} - gx \in \operatorname{ann}(G' + WZ^{[j]}) = I'$.

Suppose $wz^t - h' \in I'$ for some $t < \beta + 1$ and $h' \notin J'$. Then

$$\begin{aligned} 0 &= \frac{\partial(G' + WZ^{[j]})}{wz^t - h'} = \frac{\partial G'}{\partial wz^t} + \frac{\partial WZ^{[j]}}{\partial wz^t} - \frac{\partial G'}{\partial h'} - \frac{\partial WZ^{[j]}}{\partial h'} \\ &= Z^{[j-t]} - \frac{\partial G'}{\partial h'} - W\frac{\partial Z^{[j]}}{\partial h'}. \end{aligned}$$

Hence $\dfrac{\partial Z^{[j]}}{\partial h'} = 0$ and $\dfrac{\partial G'}{\partial h'} = Z^{[j-t]}$. This implies that x divides h'. Let $h' = xh$ for some h. Consider $wz^{t-1} - h$,

$$\begin{aligned} \frac{\partial G + WZ^{[j-1]}}{\partial wz^{t-1} - h} &= \frac{\partial G}{\partial wz^{t-1}} + \frac{\partial WZ^{[j-1]}}{\partial wz^{t-1}} - \frac{\partial G}{\partial h} - \frac{\partial WZ^{[j-1]}}{\partial h} \\ &= Z^{[j-t]} - \frac{\partial G}{\partial h} - W\frac{\partial Z^{[j-1]}}{\partial h}. \end{aligned}$$

If $\dfrac{\partial Z^{[j-1]}}{\partial h}$ is non-zero, then h contains a pure power of z, which must be z^t. But $\dfrac{\partial G'}{\partial h'} = Z^{[j-t]}$ would then mean that G' has a term XZ^j which is not possible by our assumption on F.

Therefore $\dfrac{\partial Z^{[j-1]}}{\partial h} = 0$. Since $G = \dfrac{\partial G'}{\partial x}$ and $h' = xh$, we get $\dfrac{\partial G + WZ^{[j-1]}}{\partial wz^{t-1} - h} = 0$. Therefore $wz^{t-1} - h \in I$. This contradicts the minimality of β. This completes the proof of the claim.

Suppose $I = (I_2, J, wz^\beta - g)$ is generic and let $I' = (I_2, J', wz^{\beta+1} - gx)$. First note that if $h \in \mathfrak{n}J$, then $hx_t \in \mathfrak{n}J'$ for all t. Since $gx_t \in \mathfrak{n}J$, $gx_s x_t \in \mathfrak{n}J'$ for all $1 \leq s, t \leq n$. Therefore they are not part of minimal generating set for J'. Hence I' is generic. $\square$

We may also restate the theorem as a necessary condition as follows:
Let $I = \operatorname{ann} F = G + WZ^{[j]}$. Suppose $G = X_t G_{X_t}$ for some $1 \leq t \leq n-1$. Then $I = \operatorname{ann} F$ is generic if $\operatorname{ann} G_{X_t} + WZ^{[j-1]}$ is generic.

As a result of the above theorem, we concentrate on polynomials $F = G + WZ^{[j-1]}$ in the divided power algebra such that none of the X_i, $1 \leq i \leq n-1$ divides G and of course that G has no term containing $Z^{[j-1]}$.

The converse of the above theorem is not true in general as we can see in the Example 4.2.

We remark that it is not always the case that we can achieve 'genericity' by multiplying by an X_i even if one of gx_j is a non-minimal generator for the corresponding ideal J. See the examples in the last section. However, it is an interesting question, whether if $F = G + WX_n^{[j-1]} \in k_{DP}[X_1, \ldots X_n, W]$ is not generic, does there exist a suitable power of $X_i, 1 \leq i \leq n-1$ multiplying G by which will result in a generic ideal, better yet, does there exist a suitable one form $L(X_1, \ldots X_{n-1})$ so that $I' = \operatorname{ann}(LG + WX_n^{[j]})$ is such that $\mu(I') = n + 1 + \mu(I') \cap k[x_1, \ldots, x_n]$?

2. Embedding Dimension Four

In embedding dimension 4, we can get a stronger characterization. We now let $n = 3$ so that $R = k[w, x, y, z]$.

Notation 2.1. *For the rest of the paper, we set*

$$\begin{aligned} F &= G_0 + X^{[a_1]}G_1 + \cdots + X^{[a_l]}G_l + WZ^{[j-1]} \\ &= \sum_{t=0}^{m} c_{p-t} Y^{[p-t]} Z^{[q+t]} + X^{[a_1]} \left(\sum_{r_{1k}+s_{1k}=j-a_1} c_{r_{1k}} Y^{[r_{1k}]} Z^{[s_{1k}]} \right) \\ &\quad + \cdots + X^{[a_n]} \left(\sum_{r_{nk}+s_{nk}=j-a_n} c_{r_{nk}} Y^{[r_{nk}]} Z^{[s_{nk}]} \right) + WZ^{[j-1]}, \end{aligned}$$

where $c_p \neq 0$, $a_1 < \cdots < a_n$ and one of the $G_i's$ contain a pure power of Z.

For a polynomial $h(x_1, \ldots, x_l) \in k[x_1, \ldots, x_l]$, let $\deg_{x_i} h$ denote the highest power of x_i in h. Thus, in F as in 2.1, $\deg_X F = a_{n},$.

We first obtain a necessary condition for an ideal to be generic in terms of F. Before we prove the result, we prove a technical, but very important lemma that is needed in the proof of this theorem. This lemma will play a crucial role in the proofs of all the forthcoming characterization results as well.

Lemma 2.2. *With the notation as in 2.1, if $\deg_Y G_0 = p$ then $\operatorname{ann} G_0$ doesn't have elements of degree less than or equal to $j-p$ other than y^{p+1} when $p < j-p$.*

Proof. Clearly $y^{p+1} \in \operatorname{ann} G_0$. Suppose $\operatorname{ann} G_0$ contains a generator of degree less than q, say $g = \sum_{i=0}^{k} \alpha_i y^{r-i} z^i$ for some $0 \leq k \leq r$, where $\alpha_k \neq 0$. Then we have

$$\begin{aligned} 0 = \frac{\partial G_0}{\partial g} &= \frac{\partial c_p Y^{[p]} Z^{[q]} + c_{p-1} Y^{[p-1]} Z^{[q+1]} + \ldots + c_{p-m} Y^{[p-m]} Z^{[q+m]}}{\partial \alpha_r y^r + \alpha_{r-1} y^{r-1} z + \cdots + \alpha_k y^{r-k} z^k} \\ &= \alpha_k c_p Y^{[p-r+k]} Z^{[q-k]} + \text{terms in Y of degree less than } p-r+k. \end{aligned}$$

Since $c_p \neq 0$, $\alpha_k = 0$, which is a contradiction. Hence the assertion follows. □

Theorem 2.3. *With the notation as in 2.1, if I is generic, then either $a_1 \leq \deg_Y G_0$ or $\deg_Y G_0 \leq \max\{\deg_Y G_i \ : \ i = 1, \ldots, n\}$.*

Proof. We first show that if $f \in \operatorname{ann} F$ with $\deg f < a_1$, then either $f \in \operatorname{ann} G_0 \cap \operatorname{ann} WZ^{[j-1]} \cap \operatorname{ann}(\sum_{i=1}^{n} X^{[a_i]} G_i)$ or $f \in \operatorname{ann}(G_0 + WZ^{[j-1]}) \cap \operatorname{ann}(\sum_{i=1}^{n} X^{[a_i]} G_i)$. For if $f \in \operatorname{ann} F$, then

$$0 = \frac{\partial F}{\partial f} = \frac{\partial G_0}{\partial f} + \sum_{i=1}^{n} \frac{\partial X^{[a_i]} G_i}{\partial f} + \frac{\partial W Z^{[j-1]}}{\partial f}.$$

Therefore

$$\frac{\partial G_0}{\partial f} + \frac{\partial W Z^{[j-1]}}{\partial f} = -\sum_{i=1}^{n} \frac{\partial X^{[a_i]} G_i}{\partial f}.$$

Since G_0 is a polynomial in Y and Z, the term on the left hand side of the above equality does not involve X. Since $\deg f < a_1$ we get that $\sum_{i=1}^{n} \frac{\partial X^{[a_i]} G_i}{\partial f} = 0$ and $\frac{\partial (G_0 + W Z^{[j-1]})}{\partial f} = 0$. We now proceed to the proof of the theorem. We show that if $a_i > p > r_{ik}$ for every i and k, where $p = \deg_Y G_0$, then I is not generic. We do this by considering the following two cases:

Case 1. Suppose $p \leq j - p$. By Lemma 2.2, y^{p+1} is a minimal generator for $\operatorname{ann} G_0$ since all other generators have degree bigger than $j-p$. Since $p > r_{ik}$ for all i, k, it can be seen that $y^{p+1} \in \operatorname{ann} G_0 \cap \operatorname{ann} WZ^{[j-1]} \cap \operatorname{ann} X^{[a_i]} G_i$ for all i. We then conclude that y^{p+1} is a minimal generator for $\operatorname{ann} F$.

On the other hand, note that $f = wz^{p-1} - y^p \in \operatorname{ann} F$. Since the degree of $f = p < a_1$

and contains a term in w, we get $f \in \operatorname{ann}(G_0 + WZ^{[j-1]}) \cap \operatorname{ann}(\sum_{i=1}^{l} X^{[a_i]}G_i)$. We show that f is a minimal generator. Suppose $wz^\beta - g \in I$ for some $\beta < p-1$. It implies that $y^p - z^{p-1-\beta}g \in I$ which is impossible since y^{p+1} is a minimal generator. Therefore, $wz^\beta - g \notin I$ for any $\beta < p-1$ so that f is a minimal generator for I. Since y^{p+1} is a minimal generator, I is not generic.

Case 2. Suppose $p > j-p$. In that case $wz^{p-1} - y^p \in \operatorname{ann} F$, but might not be a minimal generator for I. Let $f = wz^{\beta-1} - g(x,y,z)$ be a minimal generator for I, with $\beta \leq p$. We first show that we can replace $g(x,y,z)$ by $g(y,z)$. We write $g(x,y,z) = g_1(y,z) + xg_2(x,y,z)$. Then,

$$\frac{\partial F}{\partial f} = -\frac{\partial G_0}{\partial g_1(y,z)} - \sum_{i=1}^{n} \frac{\partial X^{[a_i]}G_i}{\partial g_1(y,z) + xg_2(x,y,z)} + \frac{\partial WZ^{[j-1]}}{\partial wz^{\beta-1}} = 0.$$

Since $a_i > p \geq \beta$, we get $\frac{\partial G_0}{\partial g_1} = Z^{[j-\beta]}$ and $\displaystyle\sum_{i=1}^{n} \frac{\partial X^{[a_i]}G_i}{\partial g_1 + xg_2} = 0$. Therefore, either $g_1 = y^p$ or the degree of g_1 is strictly less than p. But in both cases and by Lemma 2.2, the degree of g_1 is greater than $j-p$. On the other hand, the degree of G_i is at most $j-p-1$. It follows that $\frac{\partial X^{[a_i]}G_i}{\partial g_1} = 0$ for all i, and hence $xg_2 \in J$. So $g(x,y,z)$ can be replaced by $g(y,z)$.

We know that $\frac{\partial F}{\partial g(y,z)} = Z^{[j-\beta]}$ and $yg(y,z) \in \operatorname{ann} F$. In fact, $yg(y,z) \in \operatorname{ann} G_0 \cap \operatorname{ann}(WZ^{[j-1]}) \cap \operatorname{ann}(\sum_{i=1}^{n} X^{[a_i]}G_i)$. As in the case of the proof of Lemma 4.2 in [3], it can be seen that $\operatorname{ann} G_0$ is minimally generated in $k[y,z]$ by a regular sequence $(y\delta(y,z), \theta(y,z))$ with $\theta(y,z) = z^t + \theta_1(y,z)$. By Lemma 2.2, the degrees of $y\delta(y,z)$ and $\theta(y,z)$ are at least $j-p+1$ and the degrees of the $G_i's$ are at most $j-p-1$. It follows that $y\delta(y,z)$ and $\theta(y,z) \in \operatorname{ann} G_0 \cap \operatorname{ann}(\sum_{i=1}^{n} X^{[a_i]}G_i)$. But $\theta(y,z) \notin \operatorname{ann} WZ^{j-1}$. Therefore $\theta(y,z) \notin \operatorname{ann} F$, but $y\theta(y,z) \in \operatorname{ann} F$. Hence, $y\delta(y,z)$ and $y\theta(y,z)$ are minimal generators for $\operatorname{ann} F$.

We now show that $yg(y,z)$ is a minimal generator for I and can be chosen to be $y\delta(y,z)$. Suppose $yg(y,z)$ is not minimal then $yg(y,z) = f_1(y,z)y\delta(y,z) + f_2(y,z)y\theta(y,z)$. It implies that $g(y,z) = f_1(y,z)\delta(y,z) + f_2(y,z)\theta(y,z)$. Consider

$$\frac{\partial G_0}{\partial g(y,z)} = \frac{\partial G_0}{\partial f_1(y,z)\delta(y,z) + f_2(y,z)\theta(y,z)} = \frac{\partial G_0}{\partial f_1(y,z)\delta(y,z)} = Z^{[j-\beta]}.$$

We can choose $g(y,z) = f_1(y,z)\delta(y,z)$. If $f_1(y,z)$ is a constant then we are done. Otherwise $f_1(y,z) = cz^u$ since $y\delta(y,z)$ is in J. On the other hand, $\delta(y,z)$ cannot have a pure power of z, otherwise $g(y,z)$ will have a pure power of z and will not belong to I. But

$\frac{\partial G_0}{\partial\delta(y,z)} = Y\frac{\partial G_0}{\partial y\delta(y,z)} + \alpha Z^{[\gamma]} = \alpha Z^{[\gamma]}$ with $\alpha \neq 0$ and the multiplication is the usual polynomial multiplication. It implies that $f' = wz^{\beta-1-u} - \delta(y,z)$ is a minimal generator for I which contradicts the minimality of f. Hence, $g(y,z) = c\delta(y,z)$ and $yg(y,z)$ is minimal. Hence I is not generic. $\square$

In [3], it was proved that if $F = X^{[a_1]}G_1 + WZ^{[j-1]}$ with $a_1 \geq 1$, then I is generic. Example 4.8 shows that this property does not hold if $F = G_0 + X^{[a]}G_1 + WZ^{[j-1]}$. Let

$$\begin{aligned} F &= G_0 + X^{[a]}G_1 + WZ^{[j-1]} \\ &= \sum_{r=0}^{m} c_{p-r}Y^{[p-r]}Z^{[q+r]} + X^{[a]}\left(\sum_{r_i+s_i=j-a}\alpha_{r_i}Y^{[r_i]}Z^{[s_i]}\right) + WZ^{[j-1]}. \end{aligned}$$

If we suppose that $a > j-a \geq p$, then we get an improved version of Theorem 2.3 in this case.

Theorem 2.4. *Let F be as above and suppose $\deg_X F > \deg G_1 \geq \deg_Y G_0$. Then I is generic if and only if $\deg_Y G_1 = p$.*

Proof. Suppose $\deg_Y G_1 = p$. We write $F = c_p^{-1}Y^{[p]}Z^{[a+s]} + \cdots + c_2Y^{[2]}Z^{[a+s+p-2]} + X^{[a]}(\alpha_pY^{[p]}Z^{[s]} + \cdots + \alpha_0Z^{[p+s]}) + WZ^{[j-1]}$ with c_p, α_p and $\alpha_0 \neq 0$.
We first show that $wz^{p+s} - c_p^{-1}y^pz^{s+1}$ is a minimal generator for I. Clearly, $wz^{p+s} - c_p^{-1}y^pz^{s+1} \in I$. Suppose there exists $wz^\beta - g \in I$ for some $\beta < p+s$. As in the proof of Theorem 2.3, we can assume that g is a function of y and z. Then we have

$$\begin{aligned} 0 &= \frac{\partial F}{\partial wz^\beta - g} \\ &= \frac{\partial G_0 + X^{[a]}G_1 + WZ^{[j-1]}}{\partial wz^\beta - g}. \end{aligned}$$

Therefore

$$\frac{\partial G_0 + X^{[a]}G_1}{\partial g} = Z^{[j-1-\beta]} = Z^{[a]}.$$

Since G_0 does not involve X and g has degree $p+s < a$, we get $\frac{\partial G_0}{\partial g} = Z^{[a]}$ and $\frac{\partial G_1}{\partial g} = 0$. Therefore, there exists $0 \leq i \leq p-2 < a$ with $c_{p-i} \neq 0$ such that $g = c_{p-i}^{-1}y^{p-i}z^{s+i} + g_1(y,z)$. If $i \neq 0$, then $\frac{\partial G_0}{\partial g} = c_p.c_{p-i}^{-1}Y^{[i]}Z^{[a-i]} + \cdots + Z^{[a]} + \cdots$ and g_1 does not annihilate $G_0 - c_{p-i}Y^{[p-i]}Z^{[a+s+i]}$ by Lemma 2.2, which is a contradiction. Hence $i = 0$ and $g = c_p^{-1}y^pz^s + g_1(y,z)$. Therefore, $\frac{\partial G_0}{\partial g} = Z^{[a]}$ and $\frac{\partial X^{[a]}G_1}{\partial g} = c_p^{-1}\alpha_pX^{[a]}$. This implies that there exists a term $g_2(y,z)$ in $g_1(y,z)$ in y and z of total degree $p+s$ such that $\frac{\partial X^{[a]}G_1}{\partial g_2} = -c_p^{-1}\alpha_pX^{[a]}$. Again $\frac{\partial G_0 - c_pY^{[p]}Z^{[a+s]}}{\partial g_2} \neq 0$ by Lemma 2.2. Continuing in this manner, we see that $wz^\beta - g$ does not annihilate F, which is a contradiction. Therefore

$p+s$ is the minimum exponent β of z such that $wz^\beta - g \in I$ and hence $wz^{p+s} - c_p^{-1}y^pz^{s+1}$ is a minimal generator in I.

Since $y^{p+1} \in I$, $y \cdot y^pz^{s+1}$ is not a minimal generator of I. Note also that if j is the smallest integer so that $\alpha_{p-j} \neq 0$, then $x(\alpha_{p-j}y^pz^s - \alpha_p y^{p-j}z^{s+j}) \in I$. Therefore, $x \cdot y^pz^{s+1}$ is also not a minimal generator of I. Hence I is generic.

We now prove the converse. We know by Theorem 2.3 that if $a > p$ and I is generic, then $\alpha_{p+i} \neq 0$ for some $i \geq 0$. With the extra assumption that F has only two terms and $a > j-a$, we show that $\alpha_p \neq 0$ and $\alpha_{p+i} = 0$ for $i > 0$. So it suffices to show that if there exists an $i > 0$ such that $\alpha_{p+i} \neq 0$, then I is not generic. Suppose $\alpha_{p+i} \neq 0$ for some i. Let

$$F \quad = \quad c_pY^{[p]}Z^{[a+s]} + \cdots + c_2Y^{[2]}Z^{[a+s+p-2]} + X^{[a]}\left(\sum_{m=0}^{p+r} \alpha_m Y^{[m]}Z^{[p+s-m]}\right) + WZ^{[j-1]},$$

where $\alpha_{p+i} \neq 0$ for some $i > 0$. It is clear that $wz^{p+s} - {c_p}^{-1}y^pz^{s+1} \in I$. If it is not a minimal generator for I, then there exists $\beta < p+s$ such that $wz^\beta - g(y,z)$ is minimal. Then we have

$$\frac{\partial F}{\partial wz^\beta - g(y,z)} = Z^{[j-1-\beta]} - \frac{\partial G_0}{\partial g} - X^{[a]}\frac{\partial G_1}{\partial g} = 0.$$

Since $a > p+s$, we get $\frac{\partial G_1}{\partial g} = 0$ and $\frac{\partial G}{\partial g} = Z^{[j-1-\beta]}$. We first claim that $g(y,z) = {c_p}^{-1}y^pz^q + g_1(y,z)$, where $p+q = \beta+1$ and the exponents of y in $g_1(y,z)$ are greater than $p+1$. Let $g(y,z) = c_{p-i}^{-1}y^{p-i}z^q + g_1(y,z)$, with $i \geq 0$ and the exponents of y in g_1 strictly bigger than $p-i$. Then, $\frac{\partial(G_0 - c_{p-i}Y^{[p-i]}Z^{[a+s+i]})}{\partial g(y,z)} = 0$. But by Lemma 2.2, $\operatorname{ann}(G_0 - c_{p-i}Y^{[p-i]}Z^{[a+s+i]})$ doesn't contain generators of degree less than $a+s$. This implies that $i = 0$. Hence $g(y,z)$ is of the required form.

Let q be the minimal power of z such that $wz^\beta - c_p^{-1}y^pz^q - g_1(y,z)$ is a minimal generator for I with $g_1 \in \operatorname{ann} G$ and $c_p^{-1}y^pz^q + g_1 \in \operatorname{ann} G_1$. We show that $f = y(c_p^{-1}y^pz^q + g_1(y,z))$ is a minimal generator for I. Suppose not, then f can be obtained from a combination of a generator of the form $y^{p+1}z^{q-k} + g_2(y,z)$ and other generators of I. Note that $y^{p+1}z^{q-k} + g_2(y,z) \in \operatorname{ann} G_0$ and hence belongs to $\operatorname{ann} G_1$. We have also seen that $c_p^{-1}y^pz^q + g_1(y,z)) \in \operatorname{ann} G_1$. It is clear that neither of the generators $y^{p+1}z^{q-k} + g_2(y,z)$ and $c_p^{-1}y^pz^q + g_1(y,z)$ can be obtained from each other. We then study the minimal generators of $\operatorname{ann} G_1$. We know that $\operatorname{ann} G_1$ is minimally generated by a regular sequence of the form $(y\delta(y,z), \theta(y,z))$ with $\theta(y,z) = z^t + \theta_1(y,z)$. For these generators to be in I, no pure power of z of degree less than j and no terms with degree of y less than p can appear in any of them. In that case, $y\delta(y,z)$ and $\theta(y,z)$ will be multiplied by the

appropriate power of y to get rid of the power of z and have all powers of y in their terms greater or equal than $p+1$. Since both generators $y^{p+1}z^{q-k}+g_2(y,z)$ and $c_p^{-1}y^pz^q+g_1(y,z)$ are independent in $\operatorname{ann} G_1$, $y^{p+1}z^{q-k}+g_2(y,z)$ and $y(c_p^{-1}y^pz^q+g_1(y,z))$ are independent in I. Hence f is minimal and I is not generic. □

Example 4.9 shows that statement of Theorem 2.4 does not hold if $p<a<p+s$.

Now, we will consider the general case in codimension four. Let F be as in Notation 2.1. We now obtain a generalization of Theorem 2.4. Along with Notation 2.1 we further assume that G_n has a pure power of Z (in other words, Y does not divide G_n). We first prove a necessary condition for an ideal to be generic.

Theorem 2.5. *Let F be as in Notation 2.1 with $j-a_n<a_1$, Y not dividing G_n and $\deg_Y G_i<\deg_Y G_0$ for all $i=1,\dots,n-1$. If I is generic, then $\deg_Y G_n=\deg_Y G_0$.*

Proof. By Theorem 2.3, it suffices to show that if there exists an k such that $r_{ik}>p$ in G_n, then I is not generic. It is clear that $wz^{p+s}-c_p^{-1}y^pz^{s+1}\in I$. Suppose it is not a minimal generator for I, then there exists $\beta<p+s$ such that $wz^\beta-g(y,z)$ is minimal. Since

$$0=\frac{\partial F}{\partial wz^\beta-g(y,z)}=Z^{[j-1-\beta]}-\frac{\partial G}{\partial g}-\sum_i\frac{\partial X^{[a_i]}YG_i}{\partial g}-X^{[a_n]}\frac{\partial G_n}{\partial g}$$

we get $\frac{\partial G}{\partial g}=Z^{[j-1-\beta]}$ and the rest is zero.
As in the case of the proof of Theorem 2.4, we can see that $g(y,z)=c_p^{-1}y^pz^q+g_1(y,z)$ with $p+q=\beta+1$ and $g_1(y,z)$ having all powers of y greater than $p+1$. We note that $g_1\in\operatorname{ann}G_0$ and $c_p^{-1}y^pz^q+g_1\in\operatorname{ann}G_n\cap\operatorname{ann}G_i$ for all $i=1,\dots,n-1$ in that case. It can be seen that $f=y(c_p^{-1}y^pz^q+g_1(y,z))$ is a minimal generator for I. For, if not, then f can be expressed as a linear combination of other generators of I, one of which is of the form $y^{p+1}z^{q-k}+g_2(y,z)$. Note that $y^{p+1}z^{q-k}+g_2(y,z)\in\operatorname{ann}G_0\cap(\cap_i\operatorname{ann}G_i)$ and $c_p^{-1}y^pz^q+g_1(y,z))\in\cap_i\operatorname{ann}G_i$. It is clear that neither of the generators $y^{p+1}z^{q-k}+g_2(y,z)$ and $c_p^{-1}y^pz^q+g_1(y,z)$ can be obtained from each other.

Again, looking at the minimal generators of $\operatorname{ann}G_n$, we know they are a regular sequence of the following form $(y\delta(y,z),\theta(y,z))$ with $\theta(y,z)=z^t+\theta_1(y,z)$. For these generators to be in I, they must be multiplied by the appropriate power of y to get rid of the power of z and have all powers of y in their terms greater or equal than $p+1$. Since both generators $y^{p+1}z^{q-k}+g_2(y,z)$ and $c_p^{-1}y^pz^q+g_1(y,z)$ are independent in $\operatorname{ann}G_n$, $y^{p+1}z^{q-k}+g_2(y,z)$ and $y(c_p^{-1}y^pz^q+g_1(y,z))$ are independent in I. Hence f is minimal and I is not generic. □

Examples 4.10 and 4.11 show that the converse of Theorem 2.5 is not true in general. These examples suggest that we have to add more conditions on the exponents to obtain a converse of Theorem 2.5. We prove the converse with some extra assumptions.

Theorem 2.6. *Suppose* $\deg_Y G_n = p$, $\deg_Z G_n = p+s$, $\deg G_n < a_1$ *and* $\deg_Y G_i < \deg_Y G_0$ *for all* $i = 1, \dots, n-1$. *If* F *satisfies one of the two conditions below, then* I *is generic.*

(1) $\deg_Y G_i < \deg_Y (G_n - Y^p Z^s)$;
(2) $\deg_Z G_i < \deg_Z G_n$ *for all* $i < n$.

Proof. We first show that $wz^{p+s} - c_p^{-1} y^p z^{s+1}$ is a minimal generator for I. Suppose not, let $wz^\beta - g \in I$, where $\beta < p+s$. Then

$$\begin{aligned} 0 &= \frac{\partial F}{\partial wz^\beta - g} \\ &= \frac{\partial [G_0(Y,Z) + X^{[a_1]} G_1 + \cdots + X^{[a_n]} G_n + WZ^{[j-1]}]}{\partial wz^\beta - g}. \end{aligned}$$

Therefore
$$\frac{\partial [G_0(Y,Z) + X^{[a_1]} G_1 + \cdots + X^{[a_n]} G_n]}{\partial g} = Z^{[j-1-\beta]}$$

Since $a_i > j - a_n = p+s$, g may be assumed to be a function of y and z. Hence we get
$$\frac{\partial G_0(Y,Z)}{\partial g(y,z)} = Z^{[j-1-\beta]} \quad \text{and} \quad \frac{\partial X^{[a_i]} Y G_i(Y,Z)}{\partial g(y,z)} = 0.$$

By Lemma 2.2, $\operatorname{ann} G_0(Y,Z)$ does not contain generators of degree less than $a_n + s = j - p$. Therefore $g = c_p^{-1} y^p z^s + g_1$ for some polynomial g_1. But in that case, we have $\frac{\partial X^{[a_n]} G_n(Y,Z)}{\partial c_p^{-1} y^p z^s} = \alpha_p c_p^{-1} X^{[a_n]}$ and hence g_1 must contain a term g_2 with the power of y less than p such that $\frac{\partial X^{[a_n]} G_n(Y,Z)}{\partial g_2} = -X^{[a_n]}$. Continuing in this manner, we see that $wz^\beta - g$ can not annihilate F, which is a contradiction. Therefore $wz^\beta - g \notin I$ for $\beta < p+s$ and hence $wz^{p+s} - c_p^{-1} y^p z^{s+1}$ is a minimal generator.

Note that since $y^{p+1} \in I$, $y \cdot y^p z^{s+1}$ is not a minimal generator. Also we have in case (1) that $x(\alpha_p y^{p-k} z^{s+k} - \alpha_{p-k} y^p z^s) \in I$ and in case (2), $x(\alpha_p z^{s+p} - \alpha_0 y^p z^s) \in I$, which implies that $xy^p z^s$ is not a minimal generator of I. Hence I is generic. $\square$

3. Hilbert Functions of I and J

In this section we compare the Hilbert functions of I and $J = I \cap k[x,y,z]$. It can be noted that these results are independent of whether the ideal is generic or not.

Proposition 3.1. *Let $R = k[x, y, z, w]$, $F = G + WZ^{[j-1]}$, $I = \operatorname{ann}(F)$ and $J = I \cap k[x, y, z]$. Then $H_{R/I} - H_{R'/J} = [0, 1, \ldots, 1, 0, \ldots]$, where the last 1 occurs at the degree β.*

Proof. Let $I = (I_2, J, wz^\beta - g)$, where $g \in k[x, y, z] \setminus J$. Clearly $H_{R/I}(0) - H_{R'/J}(0) = 0$ and $H_{R/I}(1) - H_{R'/J}(1) = 1$. Note that $I_2 = \langle wx, wy, w^2 \rangle$ and J has generators of degree at least 3. Therefore $H_{R/I}(2) = 10 - 3 = 7$ and $H_{R'/J}(2) = 6$. For $3 \leq n \leq \beta$, I_n, as a k-vector space, is generated by I_2R_{n-2} and J_n. If a monomial $m_1 \in [R/I]_n$ is a k-basis element, then either $m = wz^{n-1}$ or w does not divide m and m is a k-basis element of $[R'/J]_n$. Therefore, $\dim_k[R/I]_n = \dim_k[R'/J]_n + 1$ for all $3 \leq n \leq \beta$. For $n = \beta + 1$, one can see that $wz^\beta - g \in I$ and $g \notin I$ and hence $g \notin J$. Moreover, for $n \geq \beta + 1$, $wz^n = z^{n-\beta-1}g \pmod{I}$ and $z^{n-\beta-1}g \in I$ if and only if $z^{n-\beta-1}g \in J$. Therefore, $H_{R/I}(n) - H_{R'/J}(n) = 0$ for $n \geq \beta + 1$. Hence the assertion is proved. □

It is known that height three Gorenstein ideal in $k[x, y, z]$ is unimodal, see [7]. As a corollary of the previous result, we obtain a class of Artinian level algebras of embedding dimension three, namely type two and at least one of the socle elements is a pure power of a one form, having unimodal Hilbert function.

Corollary 3.2. *If $J \subset R' = k[x, y, z]$ is an ideal such that $J = \operatorname{ann}(G, Z^{[\deg G-1]})$ for some polynomial G in the divided power algebra $k_{DP}[X, Y, Z]$, then the Hilbert function of J is unimodal.*

Proof. Let J be an ideal of the given form. Let $I = G + WZ^{[j-1]}$. Then it can be seen that $J = I \cap R'$. From Proposition 3.1, it follows that $H_{R'/J} = H_{R/I} - [0, 1, \ldots, 1, 0..]$, where the last 1 occurs at the degree β. Since I is a Gorenstein ideal with initial degree 2, $H_{R/I}$ is unimodal, [6, Theorem 3.1]. Therefore $H_{R'/J}$ is unimodal. □

4. Examples

In this section we provide some examples to illustrate our results. We follow the notation that was set earlier, i.e., given a homogeneous form F in the divided power algebra $k_{DP}[W, X_1, \ldots, X_n]$, $I = \operatorname{ann}(F) \subset k[w, x_1, \ldots, x_n]$ and $J = I \cap k[x_1, \ldots, x_n]$. The following is an example of Theorem 1.2.

Example 4.1. *Let $F' = G + WZ^{[9]} = X^{[8]}Y^{[2]} + X^{[3]}Y^{[3]}Z^{[4]} + X^{[2]}YZ^{[7]} + WZ^{[9]}$ and $F = \frac{\partial^2 G}{\partial X^2} + WZ^{[7]} = X^{[6]}Y^{[2]} + XY^{[3]}]Z^{[4]} + YZ^{[7]} + WZ^{[7]}$. If $I' = \operatorname{ann}(F')$ and $I = \operatorname{ann}(F)$, then it can be seen that $\mu(I') = 13$ and $\mu(J') = 9$ and $\mu(I) = 9$ and $\mu(J) = 5$.*

The following example shows that the converse of Theorem 1.2 is not true in general.

Example 4.2. *Let* $I = \operatorname{ann}(Y^{[5]}Z^{[8]} + Y^{[4]}Z^{[9]} + X^{[5]}(Y^{[6]}Z^{[2]} + Y^{[7]}Z + Y^{[8]} + Z^{[8]}) + WZ^{[12]})$ *and* $I' = \operatorname{ann}(XY^{[5]}Z^{[8]} + XY^{[4]}Z^{[9]} + X^{[6]}(Y^{[6]}Z^{[2]} + Y^{[7]}Z + Y^{[8]} + Z^{[8]}) + WZ^{[13]}))$. *Then* $\mu(I) = 11$, $\mu(J) = 8$, $\mu(I') = 13$ *and* $\mu(J') = 9$. *Therefore* I' *is generic, but* I *is not.*

The following is an example where $\operatorname{ann}(G + WZ^{[j-1]})$ is not generic and $\operatorname{ann}(X^tG + WZ^{[t+j-1]})$ is also not generic for some t.

Example 4.3. *Let* $F = G + WZ^{[12]} = Y^{[3]}Z^{[10]} + Y^{[4]}Z^{[9]} + Y^{[5]}Z^{[8]} + Y^{[2]}Z^{[11]} + X^{[6]}(Y^{[4]}Z^{[3]} + Y^{[2]}Z^{[5]} + Z^{[7]}) + X^{[7]}(Y^{[3]}Z^{[3]} + YZ^{[5]} + Z^{[6]}) + WZ^{[12]}$. *It can be seen that* $I = \operatorname{ann}(F)$ *is not generic. Moreover,* $\operatorname{ann}(X^tG + WZ^{[12+t]})$ *is not generic for* $t = 1, 2, 3$ *and that* $I = \operatorname{ann}(X^4G + WZ^{[16]})$ *is generic. In a similar manner, one can see that* $\operatorname{ann}(Y^tG + WZ^{[12+t]})$ *is also not generic for* $t = 1, 2, 3, 4$ *and that* $\operatorname{ann}(Y^5G + WZ^{[17]})$ *is generic.*

The following example, in codimension five, shows that $I = \operatorname{ann}(G + WZ^{[j-1]}) \subset k[w, t, x, y, z]$ need not be generic but $I = \operatorname{ann}(TG + WZ^{[j]})$ is generic.

Example 4.4. *Let* $I = \operatorname{ann}(Y^{[3]}Z^{[9]} + Y^{[4]}Z^{[8]} + T^{[3]}Z^{[9]} + X^{[3]}(Y^{[4]}Z^{[4]}T + Z^{[9]} + + WZ^{[11]})$ *and* $I' = \operatorname{ann}(XY^{[3]}Z^{[9]} + XY^{[4]}Z^{[8} + XT^{[3]}Z^{[9]} + X^{[4]}(Y^{[4]}Z^{[4]}T + Z^{[9]} + WZ^{[12]}))$. *Then* $\mu(I) = 14$, $\mu(J) = 10$, $\mu(I') = 16$ *and* $\mu(J') = 11$. *Here,* I *is not generic as* t^4 *is a minimal generator of* J. *But* I' *is generic for* t^4 *is no longer the obstruction, for* $wz^3 - xt^3$ *is the minimal generator of* I'.

The following is an examples of an ideal which is not generic when $a > p$ and $p > r_{ik}$.

Example 4.5. *If* $F = Y^{[10]}Z^{[5]} + Y^{[9]}Z^{[6]} + Y^{[6]}Z^{[9]} + X^{[11]}(Y^{[2]}Z^{[2]} + Z^{[4]}) + WZ^{[14]}$, *then* $\mu(I) = 9$ *and* $\mu(J) = 6$. *Here* $wz^7 - y^8 + y^7z - y^6z^2 + y^5z^3 - yz^7$ *is a minimal generator for* I. *It can also be seen that* $\operatorname{ann} G$ *is minimally generated in* $k[y, z]$ *by* $(\theta, \delta_1) = (y^8 + y^7z - y^5z^3 - 2y^4z^4 - y^3z^5 + y^2z^6 + 3yz^7 + 3z^8, y^7z^2 - y^4z^5 - y^3z^6 + yz^8 + 2z^9)$. *Note that we may replace* δ_1 *by* $3\delta_1 + (2z + y)\theta = yg(y, z)$ *which makes* $yg(y, z)$ *a minimal generator (see the proof of Theorem 2.3).*

We have noticed that if $p \leq r_{ik}$ for some i and k, then the classification is more complicated. It does not only depend on the powers alone, but also on the coefficients as we see in the example below:

Example 4.6. *If* $I = \operatorname{ann}(Y^{[3]}Z^{[8]} + Y^{[4]}Z^{[7]} + X^{[5]}(Y^{[4]}Z^{[2]} + Y^{[3]}Z^{[3]} + Z^{[6]}) + WZ^{[10]})$ *then* $\mu(I) = 9$ *and* $\mu(J) = 5$ *whereas if* $I = \operatorname{ann}(Y^{[3]}Z^{[8]} + Y^{[4]}Z^{[7]} + X^{[5]}(2.Y^{[4]}Z^{[2]} + Y^{[3]}Z^{[3]} + Z^{[6]}) + WZ^{[10]})$ *then* $\mu(I) = 11$ *and* $\mu(J) = 8$.

Examples 4.7 and 4.8 are examples of Theorem 2.4.

Example 4.7. *Let* $F = Y^{[4]}Z^{[10]} + Y^{[3]}Z^{[11]} + X^{[8]}(Y^{[4]}Z^{[2]} + Y^{[3]}Z^{[3]} + Y^{[2]}Z^{[4]} + Z^{[6]}) + WZ^{[13]}$. *In this case,* $deg_Y G_1 = p$ *and one can see that the ideal* $I = \operatorname{ann}(F)$ *is generic with* $\mu(I) = 11$ *and* $\mu(J) = 7$.

Example 4.8. *Let* $F = Y^{[4]}Z^{[10]} + Y^{[3]}Z^{[11]} + X^{[8]}(Y^{[5]}Z + Y^{[4]}Z^{[2]} + Y^{[3]}Z^{[3]} + Y^{[2]}Z^{[4]} + Z^{[6]}) + WZ^{[13]}$. *In this case* $deg_Y G_1 > p$ *and we have* $\mu(I) = 11$ *and* $\mu(J) = 8$.

Example below shows that Theorem 2.4 doesn't work if $p \leq a$:

Example 4.9. *Let* $F = Y^{[4]}Z^{[7]} + Y^{[3]}Z^{[8]} + X^{[4]}(Y^{[4]}Z^{[3]} + Z^{[7]}) + WZ^{[10]}$. *Here,* $a < \deg_X F = 4 < \deg G_1 = 7$. *One can see that* $\mu(I) = 9$ *and* $\mu(J) = 6$. *Whereas if* $F = Y^{[4]}Z^{[7]} + X^{[4]}(Y^{[4]}Z^{[3]} + Z^{[7]}) + WZ^{[10]}$, *then* $\mu(I) = 9$ *and* $\mu(J) = 5$. *In the above examples, we have* $\deg_X F < \deg G_1$. *While the first one is generic, the second one is not. This shows that our characterization is not valid without the given hypotheses.*

The example below shows that the converse of Theorem 2.5 is not true.

Example 4.10. *It can be seen that* $\operatorname{ann}(Y^{[5]}Z^{[12]} + Y^{[4]}Z^{[13]} + X^{[8]}(Y^{[4]}Z^{[5]} + Y^{[3]}Z^{[6]} + Y^{[2]}Z^{[7]} + YZ^{[8]}) + X^{[9]}(Y^{[4]}Z^{[4]} + Y^{[3]}Z^{[5]}) + X^{[10]}(Y^{[5]}Z^{[2]} + Y^{[4]}Z^{[3]} + Y^{[3]}Z^{[4]} + Z^{[7]}) + WZ^{[16]})$ *is generic whereas* $\operatorname{ann}(Y^{[5]}Z^{[12]} + Y^{[4]}Z^{[13]} + X^{[7]}(Y^{[4]}Z^{[6]} + Y^{[3]}Z^{[7]} + Y^{[2]}Z^{[8]} + YZ^{[9]}) + X^{[9]}(Y^{[4]}Z^{[4]} + Y^{[3]}Z^{[5]}) + X^{[11]}(Y^{[5]}Z + Y^{[4]}Z^{[2]} + Y^{[3]}Z^{[3]} + Z^{[6]}) + WZ^{[16]})$ *is not. Note that in both cases* $j - a_1 < a_1$, $\deg_Y G_i < \deg_Y G_0$ *and* $\deg_Y G_3 = \deg_Y G_0 = 5$.

The example below shows that we cannot remove the conditions $\deg_Y = G_i < \deg_Y = G_0, 1 \leq i \leq n-1$ in Theorem 2.5. In F, all conditions except $\deg_Y G_1 = 6 > \deg_Y G_0 = 5$ are satisfied and $\operatorname{ann} F$ is generic and the conclusion fails where as in G all conditions except $\deg_Y = G_0 \neq \deg_Y = G_n$. Hence by the theorem 2.5, $\operatorname{ann} G$ cannot be generic.

Example 4.11. *Let* $F = Y^{[5]}Z^{[12]} + Y^{[4]}Z^{[13]} + X^{[9]}(Y^{[4]}Z^{[4]}) + X^{[10]}(Y^{[6]}Z + Y^{[2]}Z^{[5]} + Y^{[3]}Z^{[4]} + Z^{[7]}) + X^{[12]}(YZ^{[4]} + Z^{[5]}) + WZ^{[16]}$. *In this case,* $6 = r_{21} > p = 5$, $\mu(I) = 15$ *and* $\mu(J) = 11$, *which implies that* I *is generic. On the other hand, if* $G = Y^{[5]}Z^{[12]} + Y^{[4]}Z^{[13]} + X^{[9]}(Y^{[4]}Z^{[4]}) + X^{[10]}(Y^{[6]}Z + Y^{[2]}Z^{[5]} + Y^{[3]}Z^{[4]} + Z^{[7]}) + WZ^{[16]}$, *then* $\mu(I) = 15$ *and* $\mu(J) = 12$ *as predicted by Theorem 2.5.*

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